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Stationary Axisymmetric Solutions of the Jordan–Brans–Dicke Equations with Electromagnetic Sources

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Any stationary axisymmetric solution of the coupled JBD-Maxwell field equations such that $F^*_{ab}t^a\psi^b$ vanishes, can be determined from a composition of any stationary axisymmetric Einstein-Maxwell spacetime with the Weyl class of metrics.

The purpose of this work is to give a theorem on stationary axisymmetric solutions of the Jordan-Brans-Dicke field equations coupled with electromagnetic fields.

Theorem. Any stationary axisymmetric spacetime satisfying the Maxwell-Jordan-Brans-Dicke field equations

$$R_{ab}=8\pi T_{ab}\Omega^{-1}-\omega\Omega_{;a}\Omega_{;b}\Omega^{-2}-\Omega^{-1}\Omega_{;a;b},\qquad \Omega_{;a}^{;a}=0$$

$$4\pi T_{ab}=F^{s}_{a}F_{sb}+\tfrac{1}{4}g_{ab}F^{rs}F_{rs} \tag{1}$$

$$F^{ab}_{;b}=0,\qquad F^*_{ab}t^a\psi^b\equiv\left(F_{ab}+\frac{i}{2}e_{abcd}F^{cd}\right)t^a\psi^b=0$$

where t^a and ψ^a are, respectively, timelike and spacelike Killing vectors, without loss of generality, can be given by the metric

$$g=e^{-2U}\{f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2\,d\psi^2]-f(dt-W\,d\psi)^2\} \tag{2}$$

where

$$\Omega=:e^{2U},\qquad \gamma=\gamma(EM)+\gamma(J),\qquad \gamma(J)=(3+2\omega)k \tag{3}$$

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$\gamma(\mathrm{EM})$ denotes the function γ of the Einstein–Maxwell theory, $\gamma(J)$ is the contribution of the JBD field to γ, and k stands for the γ of the static Weyl metric.

The structural functions f, W, and $\gamma(\mathrm{EM})$ are determined from the Einstein–Maxwell field equations with electromagnetic vector $A_\mu(\rho, z) = \delta^\psi_\mu A_\psi + \delta^t_\mu A_t$ arising for canonical stationary axisymmetric metrics, i.e., the g from (2) with $U = 0 = k$. From the well-known Ernst (1968, 1974) equations

$$\mathscr{E} := f - |\phi|^2 - i\chi, \qquad \phi := A_t + i\mathscr{A}$$
$$f\,\Delta\mathscr{E} = (\nabla\mathscr{E} + 2\phi^*\,\nabla\phi)\,\nabla\mathscr{E}, \qquad f\,\Delta\phi = (\nabla\mathscr{E} + 2\phi^*\,\nabla\phi)\,\nabla\phi \tag{4}$$
$$\Delta = \partial_{\rho\rho} + \partial_{zz} + \rho^{-1}\partial_\rho, \qquad \nabla = i_z\partial_z + i_\rho\partial_\rho, \qquad i_z i_z = 1 = i_\rho i_\rho, \qquad i_z i_\rho = 0$$

one determines the set of functions $\{f, \chi, \mathscr{A}, A_t\}$. This set of functions allows one to integrate the equations for the remaining functions W, A_ψ, and $\gamma(\mathrm{EM})$, namely

$$\begin{aligned}
W_{,z} &= \rho f^{-2}\{\chi_{,\rho} + 2[\mathscr{A}A_{t,\rho} - A_t\mathscr{A}_{,\rho}]\}\\
W_{,\rho} &= -\rho f^{-2}\{\chi_{,z} + 2[\mathscr{A}A_{t,z} - A_t\mathscr{A}_{,z}]\}\\
A_{\psi,z} &= \rho f^{-1}\mathscr{A}_{,\rho} - WA_{t,z}\\
A_{\psi,\rho} &= -\rho f^{-1}\mathscr{A}_{,z} - WA_{t,\rho}\\
2\gamma_{,\rho} &= \{\tfrac{1}{2}f^3\rho^{-2}[(W_{,z})^2 - (W_{,\rho})^2] - f_{,\rho}/\rho + \tfrac{1}{2}[(f_{,\rho})^2 - (f_{,z})^2]f^{-1}\\
&\quad + 2[(\mathscr{A}_{,z})^2 - (\mathscr{A}_{,\rho})^2 + (A_{t,z})^2 - (A_{t,\rho})^2]\}\rho f^{-1}\\
-2\gamma_{,z} &= \{f^3\rho^{-2}W_{,\rho}W_{,z} + f_{,z}/\rho - f_{,\rho}f_{,z}f^{-1} + 4[\mathscr{A}_{,\rho}\mathscr{A}_{,z} + A_{t,\rho}A_{t,z}]\}\rho f^{-1}
\end{aligned} \tag{5}$$

Notice that these equations are integrable by virtue of (4).

The potential U should satisfy the Laplace equation

$$\Delta U = 0, \qquad U := \tfrac{1}{2}\ln\Omega \tag{6}$$

and the integrable function k is given by

$$k = \int \{2\rho U_{,z}U_{,\rho}\,dz + \rho[(U_{,\rho})^2 - (U_{,z})^2]\,d\rho\} \tag{7}$$

One recognizes in (6) and (7) the vacuum Einstein field equations for the structural functions U and k of the Weyl class of metrics, i.e., the g from (2) with $U \to 0$, $W \to 0$, $\gamma \to k$, $f \to \exp(2U)$.

Therefore, one arrives at the conclusion that any stationary axisymmetric JBD spacetime with electromagnetic sources $(F^*_{ab}t^a\psi^b = 0)$ can be obtained by a "composition" of a stationary axisymmetric Einstein–Maxwell spacetime with the Weyl class of metrics.

The proof of this theorem can be easily established in a manner similar to the demonstration of the corresponding theorem for the vacuum case (Garcia *et al.*, to appear), and so we omit the proof.

With minor modifications ($z \to it$, $t \to iz$, $W \to iW$) these results can be extended to cylindrically symmetric spacetimes with electromagnetic sources. Finally, it should be pointed out that generating solution techniques (Hauser and Ernst, 1980; Hoesenlaers and Dietz, 1984; Neugebauer and Kramer, 1969) can be used in the studied case.

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REFERENCES

Ernst, F. J. (1968). *Physical Review*, **168**, 1415.

Ernst, F. J. (1974). *Journal of Mathematical Physics*, **15**, 1409.

García, A., Bretón, N., and Hauser, I. (to appear). Stationary axisymmetric solutions in the vacuum Jordan–Brans–Dicke theory, *International Journal of Theoretical Physics*, to appear.

Hauser, I., and Ernst, F. J., (1980). *Journal of Mathematical Physics*, **21**, 1126.

Hoesenlaers, C., and Dietz, W., eds. (1984). *Solutions of Einstein's Equations: Techniques and Results*, Springer-Verlag, Berlin.

Neugebauer, G., and Kramer, (1969). D. *Annalender Physik*, **24**, 62.